1 Ebert's Hat Game

1.1 Rules of the game

Consider 3 players, Alice, Bob, and Charlie. The game-master will place a hat (either white or black) on each of their heads with equal probability. After some time has passed, all three players will simultaneously either guess their hat color or pass.

Alice, Bob, and Charlie win the game if at least one player correctly guesses their hat color and no one guesses incorrectly.

For example, if Alice guesses correctly and Bob and Charlie pass, the players win the game. However, if Alice and Bob guess correctly, but Charlie guesses incorrectly, the players lose the game. If all three players pass, they lose the game.

To summarise:

- No player can see their own hat color, but each player can see the hat color of every other player.
- No communication is allowed between players, except for a strategy-planning meeting before they walk into the room.
- Each player can either guess their hat color or pass.
- All players will guess simultaneously.
- The group wins the game if at least one player guesses correctly and no one guesses incorrectly.

1.2 Problem Statement

1. Find a strategy that results in the highest win probability.
2. *Optional*: Generalize your solution to (1) for the 4, 5, ..., n player situation.

2 Solution for the 3 player game

2.1 Two simple strategies

Lets start by figuring all the possible configurations of hats we might encounter. Since each player could have either a white (W) or a black (B) hat placed on his/her head, there are $2^3 = 8$ possible configurations.

Figure 1: Possible configurations for 3 player game

WWW WWB WBW WBB BWW BWB BBW BBB

2.1.1 Random choice

As an exercise, we can consider what happens if each player randomly picks one of the three actions: white (W), black (B), or pass (P). Since we have 3 people each with 3 possible actions, there are $3^3 = 27$ total *strategies*, each being equally likely. For any of our 8 configurations, we find 7 of the 27 possible strategies win the game. I've shown the seven winning strategies for the WBW game below, and one can verify that, due to symmetry, there are exactly 7 winning strategies for each possible configuration.

Figure 2: Winning strategies for WBW game

Alice	Bob	Charlie
W	P	P
P	B	P
P	P	W
W	B	P
W	P	W
P	B	W
W	B	W

Under random choice, we pick any one of the 27 strategies with equal probability ($\frac{1}{27}$), so the probability of picking one of the 7 winning strategies is $\frac{7}{27} \approx 26\%$.

2.1.2 Restricting to 1 guesser

We can do better than 26% by making a simple observation. If we consider strategies where Alice will guess, Bob or Charlie should always pass. Either Alice guesses right, and they win, or Alice guesses wrong, and they lose. The only way for Bob or Charlie to affect the outcome by not passing is to guess wrong and lose a game they could have won.

Consider the strategy of having Alice guess at random and Bob and Charlie always pass. This strategy wins whenever Alice guesses correctly, almost doubling our previous win percentage, up to 50%.

But can we do even better...

2.2 Optimal strategy

A key observation for this game is that no matter what strategy Alice, Bob, and Charlie come up with, they can never increase their individual chance of guessing correctly to be above 50%. It is tempting, therefore, to set the upper bound on win probability to be 50%; After all, having more than one player guess lowers our chances of winning, and any one player can only ever have a 50% chance of guessing correctly...

Lets consider the following strategy and the subsequent results of each of the 8 possible game configurations:

Figure 3: Optimal strategy for 3 player game

A player will pass if they see a white and a black hat.
A player will guess black if they see 2 white hats.
A player will guess white if they see 2 black hats.

Figure 4: 3 player game with optimal strategy

Game Configuration	Alice	Bob	Charlie	Outcome
WWW	B	B	B	Loss
WWB	P	P	B	Win
WBW	P	B	P	Win
WBB	W	P	P	Win
BWW	B	P	P	Win
BWB	P	W	P	Win
BBW	P	P	W	Win
BBB	W	W	W	Loss

We've somehow managed to increase our win percentage to 75%! Further, notice how, for each individual player, they each guess 4 times and are wrong 50% of the time. Over the 8 possible games, we make 12 total guesses, 6 of which are correct, consistent with our belief that we can never guess at better than 50%.

However, by concentrating our wrong guesses into a few cases, and by spreading our right guesses out as thin as possible, we've managed to devise a strategy that wins 75% of the time!

3 Generalizing our strategy for $N \geq 3$

3.1 Formalizing our 3 player strategy

To figure out why this strategy works, and to generalize it for an N player game, we must first convert the game into a more mathematical framework. To do this, instead of thinking of the hats as white or black, we will think of them as 0's and 1's, where white $= 0$ and black $= 1$.

3.1.1 Hamming codes

Thinking of hat colors as binary variables turns our set of possible game configurations into a *Hamming space*. Each binary string representing a single game configuration is called a *code*. Our 3 player game forms a Hamming space with $N = 3$.

Definition 3.1 (Hamming space)**.** A Hamming space of size N is the set of 2^N binary strings of length N.

Definition 3.2 (Code)**.** A code is any one of the length N binary strings that comprises a Hamming space of size N.

Figure 5: Hamming codes for the 3 player game

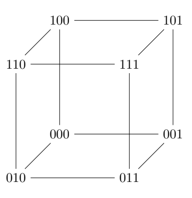
000 001 010 011 100 101 110 111

We can also define the *Hamming distance* between any two codes to be the number of places at which the codes differ. For example, codes 000 and 001 have a Hamming distance of 1. Codes 000 and 111 have a Hamming distance of 3.

Definition 3.3 (Hamming distance)**.** The Hamming distance between two codes is the number of positions at which they differ.

An elegant way to visualize our 3 player Hamming space is to consider the hypercube where $N = 3$, with nodes consisting of our 8 codes and edges connecting all codes that are a Hamming distance 1 away from each other. Then, the Hamming distance between any two codes is the number of edges one must traverse to travel between them.

Figure 6: Hypercube for $N = 3$ Hamming space



3.1.2 Covering a Hamming space

Lets designate a certain subset of these codes as *parents*. Any code that is a Hamming distance of 1 away from a parent is a *child* and forms a *family* with that parent.

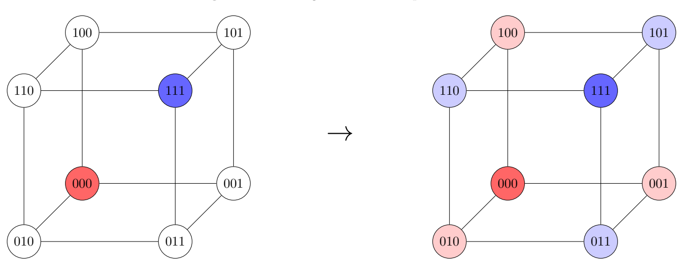
Definition 3.4 (Parent)**.** A designated code chosen from a Hamming space.

Definition 3.5 (Child)**.** A code that is a Hamming distance of 1 away from a parent.

Definition 3.6 (Family)**.** The set of a parent code and all its children.

Consider designating codes 000 and 111 as parents.

Figure 7: Choosing 000 and 111 as parents



Each parent has 3 children, and every code in the Hamming space uniquely belongs to a family. When every code belongs to at least 1 family, we call the space *covered* by the parent codes. We call this covering *minimal* when it uses the least number of parent codes possible.

Definition 3.7 (Covered)**.** A set of parent codes that result in every code in a Hamming space belonging to at least one family.

Definition 3.8 (Minimal covering)**.** A set of parent codes that covers a Hamming space with as few parents as possible.

For $N = 3$, the minimal covering consists of 2 parents. Note that our choice of parents is not unique...we could have just as easily chosen 010 and 101 as our minimal covering parents. We couldn't, however, choose any two codes at random (do you see why 010 and 001 won't work?). Finally, we could also cover this space with 3 parents, say 000, 011, and 100, but the coverage would not be minimal due to the existence of the 2 parent solution.

3.1.3 Restating our 3 player strategy using Hamming codes

Using this new framework notice that each player will have a set of two possible configurations to consider, which we will call that player's *possibilities*. Taking Alice as an example, if she sees Bob and Charlie are wearing a black and a white hat respectively, then her possibilities are 010 and 110.

Definition 3.9 (Possibilities)**.** The set of two codes that are still possible game configurations after observing the other player's hats.

Now consider the following strategy...

Figure 8: Optimal strategy using Hamming codes

1: Choose a minimal covering of the Hamming space and note the parent codes.
2: If a player's possibilities include a parent code, guess the *other* possibility.
3: If a player's possibilities do not include a parent code, pass.

By choosing the minimal covering parent codes of 000 and 111, we have exactly recreate our 3 player strategy!

Whenever the game configuration is a child code, two players will have possibilities that do not include a parent code and will pass. One player's possibilities *will* include a parent code, and they will guess consistent with the child code, winning the game.

Whenever the configuration is a parent code, *all* players will have possibilities that include a parent code, they will *all* guess consistent with their associated child code, and they will *all* be wrong, losing the game.

Figure 9: Winning conditions under optimal strategy

Win: The players will win if the game configuration is a child.
Loss: The players will lose if the game configuration is a parent.

Figure 10: Winning 3 player game: WBW

Player	Sees	Possibilities	Guess
Alice	_BW	WBW, BBW	P
Bob	W_W	**WWW**, WBW	B
Charlie	WB_	WBW, WBB	P

Figure 11: Losing 3 player game: WWW

Player	Sees	Possibilities	Guess
Alice	_WW	**WWW**, BWW	B
Bob	W_W	**WWW**, WBW	B
Charlie	WW_	**WWW**, WWB	B

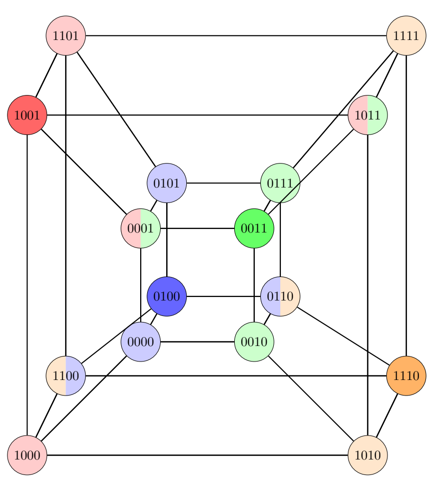
Notice that our strategy does not rely on the number of players! For any $N \geq 3$, finding the optimal strategy is equivalent to finding the minimal covering for a Hamming space of size N. We would then note the associated parent codes and follow the strategy outlined in Figure 8. We will win the game if our configuration is a child code, and we will lose the game if our configuration is a parent code.

3.2 Covering the $N = 4$ Hamming space

Now that we've reduce the base problem down to finding minimal coverings of Hamming spaces, we can attempt to find a covering for $N = 4$. We are immediately confronted with a problem. Namely, when $N = 4$ we have $2^4 = 16$ codes, but each family will have 5 members (a parent and 4 children). Therefore we can't find a perfect covering for this hypercube, since 5 does not divide 16 (for the $N = 3$ case, we could, as we had 8 codes and a family size of 4). What are we to do?

The solution is to just live with having overlapping children among the parents, and try to find the most efficient coverage available. Consider the following covering of a $N = 4$ hypercube.

Figure 12: Minimal covering for $N = 4$ hypercube



In this covering, our parent codes (1001, 0011, 0100, and 1110) share some child codes. We can confirm that this covering is minimal by noting the for $N = 4$ the maximum family size is 5 and with 3 families we could only cover 15 codes (out of a necessary 16). Thus, 4 families are needed. The four overlapping child codes occur because with 4 families, each of size 5, we have a coverage *power* of 20. Since we only need to cover 16 codes, we will overlap on 4 of them.

Lets use our strategy from Figure 8 with our covering from Figure 12 and examine a couple of games where $N = 4$.

Figure 13: Winning 4 player game: 1011

Player	Sees	Possibilities	Guess
Alice	_011	**0011**, 1011	1
Bob	1_11	1011, 1111	P
Charlie	10_1	**1001**, 1011	1
Drew	101_	1010, 1011	P

Figure 14: Losing 4 player game: 0011

Player	Sees	Possibilities	Guess
Alice	_011	**0011**, 1011	1
Bob	0_11	**0011**, 0111	1
Charlie	00_1	0001, **0011**	1
Drew	001_	0010, **0011**	1

We notice that we are still concentrating our wrong guesses when the game configuration is a parent code. However, due to the inefficiency of our covering, we are not spreading our correct guesses *as* thin as we could in the 3 player game...the inefficient winning games where 2 people guess correctly are exactly those where the child code has 2 parents!

3.3 Computing win probabilities for $N \geq 3$

We can calculate the winning probability of a game with N players by letting $P(N)$ be the number of parent codes required to minimally cover an N-dimensional Hamming space. For example, as demonstrated above, $P(3) = 2$ and $P(4) = 4$.

The probability of wining is exactly the probability of *not* having a parent code as our game configuration.

$$\text{Probability of win} = \frac{2^N - P(N)}{2^N} = 1 - \frac{P(N)}{2^N}$$

For both $N = 3$ and $N = 4$, this probability is 75%. This means that adding a fourth player didn't help us find a better strategy...we could have simply ignored the fourth team member, executed our 3 player strategy, and won just as often! I wish someone had told me before I spent an hour figuring out how to color a 4D hypercube in LaTeX...

When $N = 5$, however, we can cover our Hamming space with 7 parent codes, leading to a winning percentage of $\frac{2^5-7}{2^5} = \frac{25}{32} \approx 78\%$.

As N gets bigger and bigger, P(N) grows more slowly than 2^N, the total number of codes. Therefore our probability of winning approaches 1 (assuming the players can keep track of all those parent codes)!

$$\lim_{N\to\infty} 1 - \frac{P(N)}{2^N} = 1$$

Figure 15: Wining probabilities for $N \geq 3$

N	P(N)	Fraction of winning codes	Win probability
3	2	$\frac{6}{8}$	.75
4	4	$\frac{12}{16}$	.75
5	7	$\frac{25}{32}$	.78125
6	12	$\frac{52}{64}$	.8125
7	16	$\frac{112}{128}$	.875
8	32	$\frac{224}{256}$	.875
15	2048	$\frac{30720}{32768}$	.9375
16	4096	$\frac{61440}{65536}$	.9375
31	2^{26}	$\frac{2^{31}-2^{26}}{2^{31}}$	$1-(\frac{1}{2})^5$
2^m-1	2^{N-m}	$\frac{2^N-2^{N-m}}{2^N}$	$1-(\frac{1}{2})^m$
2^m	2^{N-m}	$\frac{2^N-2^{N-m}}{2^N}$	$1-(\frac{1}{2})^m$

3.4 Closing remarks

Notice how, whenever $N = 2^m - 1$, there exists a *perfect* covering for the Hamming space, where $P(N) = 2^{N-m}$. The first such case is when $m = 2$ and $N = 3$ using $P(N) = 2^{3-2} = 2$ parent codes. The next case is when $m = 3$ and $N = 7$ using $P(N) = 2^{7-3} = 16$ parent codes. These cases are associated with large jumps in win probability given the efficiency of the resultant minimal covering.

Also notice how, outside of numbers of the form $N = 2^m - 1$ and $N = 2^m$, there is no clear pattern for figuring out $P(N)$. As N gets larger, finding minimal coverings is a very challenging mathematical task. In fact, for values as small as $N = 10$, $P(N)$ is unknown!

If you've made it this far, thanks for reading! If you want to point out a mistake, make an insightful comment, or point me towards even cooler hat problems, please don't hesitate to reach out.

4 References

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